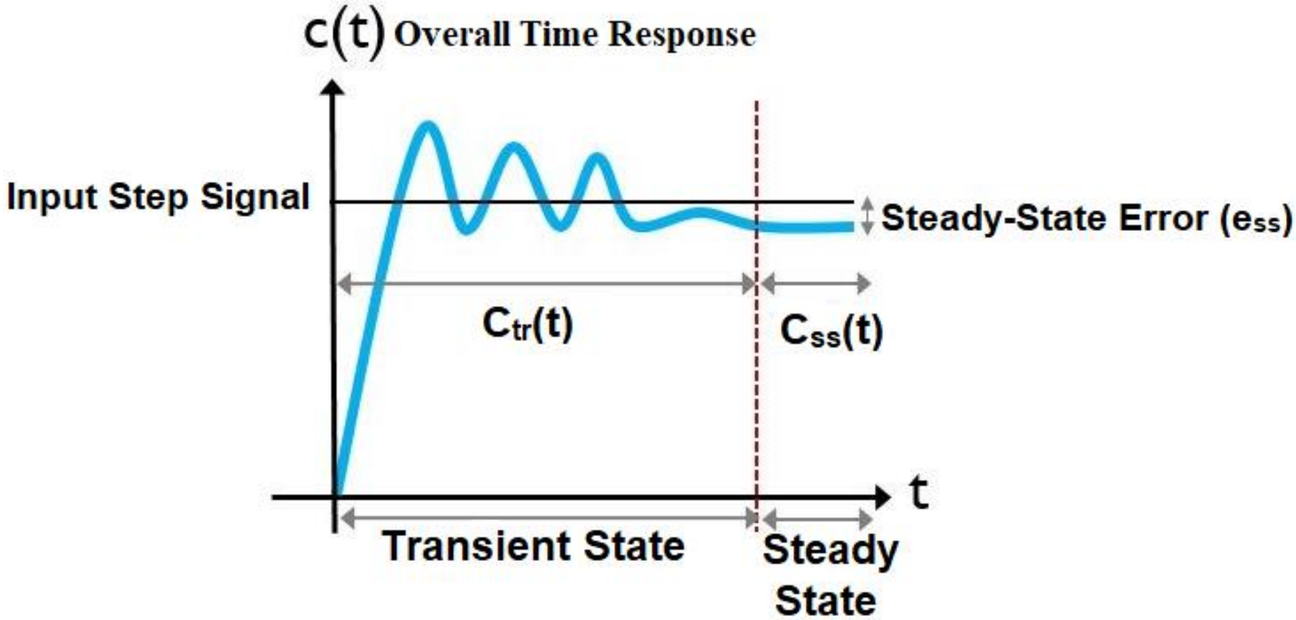
# Lecture5: Time Domain Analysis (Part 1)

## 5.1 Time Response:

The time response of a control system consists of two parts:

- ❖ Transient Response.
- ❖ Steady-State Response.

The response of control system in time domain is shown in the following figure.




- ❖ **Transient Response**: It is defined as the response of the system as the variation in output of the system before achieving the final value when excited with the input signal.

Thus, we can say that the output of the system, before the actual value, is known as a transient response. As the transient response is not the final value thus the time taken by the system to attain the desired value is known as a **transient period**.

- ❖ **Steady-State Response:** The actual value of the time response, that is achieved after the elimination of the transient response is known as a steady-state response.

Mathematically, we can write the overall time response **c(t)** as:

$$\boldsymbol{c(t) = c_{tr}(t) + c_{ss}(t)}$$

Where, $c_{tr}(t)$ is the **transient response** and $c_{ss}(t)$ is the **steady state response**.

## 5.2 Initial and Final Value Theorems:

If c(t) and C(s) is Laplace transform pairs. i.e

$$\boldsymbol{c(t) \Leftarrow (L) \Rightarrow C(s)}$$

In mathematical analysis, the **initial value theorem (IVT)** is a theorem used to relate frequency domain expressions to the time domain behavior as time approaches zero. Thus, the initial value is given by:

$$\boldsymbol{Initial\, Value = \lim_{t \to 0} c(t) = \lim_{s \to \infty} s\, C(s)}$$

In mathematical analysis, the **final value theorem (FVT)** is a theorem used to relate frequency domain expressions to the time domain behavior as time approaches infinity. Thus, the final value is given by:

$$\boldsymbol{final\, Value = \lim_{t \to \infty} c(t) = lim_{s \to 0} s\, C(s)}$$

**Example 1: Find the initial and final values of the following time response:**

$$\boldsymbol{c(t) = 10 + 5e^{-t}}$$

**Solution:**

$$Initial\, Value = \lim_{t \to 0} c(t) = \lim_{t \to 0}(10 + 5e^{-t}) = 10 + 5e^{-0} = 10 + 5 = 15$$

$$Final\, Value = \lim_{t \to \infty} c(t) = \lim_{t \to \infty}(10 + 5e^{-t}) = 10 + 5e^{-\infty} = 10 + \frac{5}{e^{\infty}} = 10 + \frac{5}{\infty} = 10$$

**Example 2: Find the initial and final values of the following function:**

$$\boldsymbol{C(s) = \frac{15\, s + 10}{s(s+1)}}$$

**Solution:**

$$Initial\, Value = \lim_{s \to \infty} s\, C(s) = \lim_{s \to \infty} s \frac{15\, s + 10}{s(s+1)} = \lim_{s \to \infty} \frac{15\, s + 10}{(s+1)}$$

**Using L'Hospital's Rule:**

If $\boldsymbol{\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \; or \; \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\pm\infty}{\pm\infty}}$

where **(a)** can be any real number, infinity or negative infinity. In these cases, we have:

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

$$Initial\ Value=\lim_{s\to\infty}\frac{15}{1}=15$$

$$Final\ Value=\lim_{s\to 0} s\,C(s)=\lim_{s\to 0} s\,\frac{15\,s+10}{s(s+1)}=\lim_{s\to 0}\frac{15\,s+10}{(s+1)}=\frac{15\,(0)+10}{(0+1)}=\frac{10}{1}=10$$

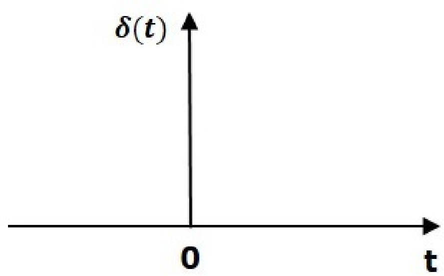
**5.3 Standard Test Signals:**

The standard input (test) signals are used to know the performance of the control systems using time response of the output. These signals are:

- **Unit Impulse Signal:**

A unit impulse signal **δ(t)** is defined as:

$$\boldsymbol{r(t)=\delta(t)\quad ,r(t)=0\ for\ t\neq 0}$$

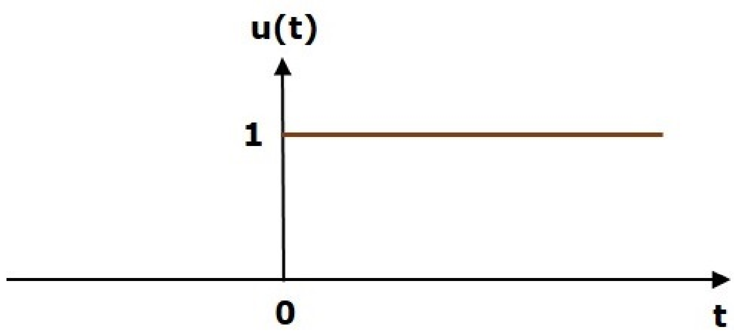



After applying Laplace transform on both the sides: **R(s) =1**.

- **Unit Step Signal:**

A unit step signal **u(t)** is defined as:

$$\boldsymbol{r(t)=u(t)=\begin{cases}1, & t\geq 0\\ 0, & t<0\end{cases}}$$

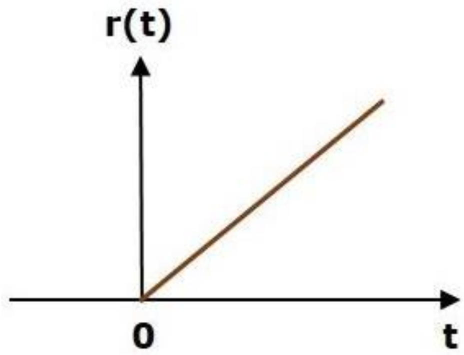



After applying Laplace transform on both the sides: $\boldsymbol{R(s)=\frac{1}{s}}$.

### Unit Ramp Signal:

A unit ramp signal **r (t)** is defined as:

$$\boldsymbol{r(t) = t\,u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}}$$

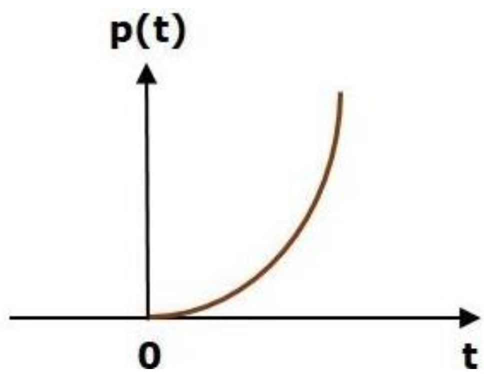



After applying Laplace transform on both the sides: $\boldsymbol{R(s)} = \frac{\boldsymbol{1}}{\boldsymbol{s^2}}$.

### Unit Parabolic Signal:

A unit parabolic signal **p(t)** is defined as:

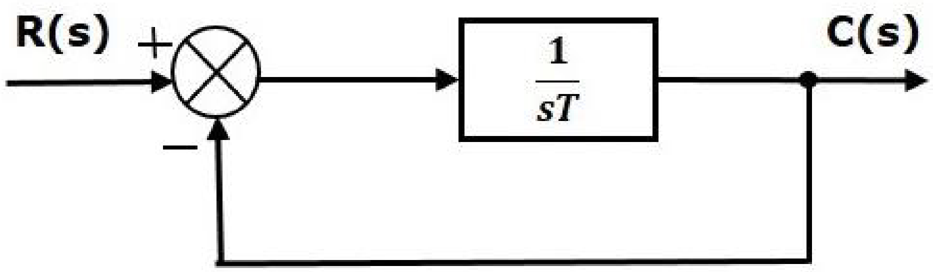
$$\boldsymbol{r(t) = p(t) = \frac{t^2}{2}\,u(t) = \begin{cases} \frac{t^2}{2}, & t \geq 0 \\ 0, & t < 0 \end{cases}}$$




After applying Laplace transform on both the sides: $\boldsymbol{R(s)} = \frac{\boldsymbol{1}}{\boldsymbol{s^3}}$.

## 5.4 First-Order Systems:

Consider the following block diagram of the closed loop control system. Here, an open loop transfer function, **1/sT** is connected with a unity negative feedback.




To get the response (output) of the first order system in the time domain, we must follow these four steps:

1) Take the Laplace transform of the input signal **r(t)** in order to get **R(s)**.
2) Substitute **G(s)** in the equation:

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Where $\frac{C(s)}{R(s)}$ is the transfer function of the closed loop control system with unity negative feedback.

3) Substitute **R(s)** value in the above equation in order to get **C(s)**.
4) Do partial fraction of **C(s)** if required, then apply inverse Laplace transform to **C(s)** in order to get **c(t)**.

The **error signal E(s)** of this system is the difference between the input and the output. So,

$$E(s) = R(s) - C(s)$$

Where, **E(s)** is the Laplace transform of the error signal **e(t)**.

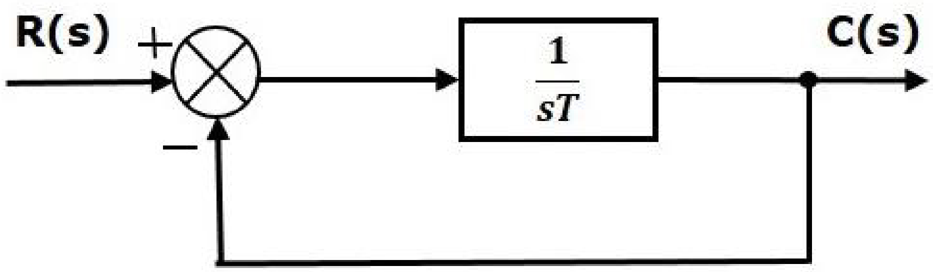
And after applying inverse Laplace transform on both the sides:

$$e(t) = r(t) - c(t)$$

Therefore, the deviation of the output of control system from desired response during steady state is known as steady state error. It is represented as **($e_{ss}$)**. We can find steady state error using the **final value theorem FVT**, as follows:

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} s\,E(s)$$

**Example 3: For the following block diagram of the closed loop control system, find the time response, initial value, final value, and steady-state error if the applied input is a unit step signal.**

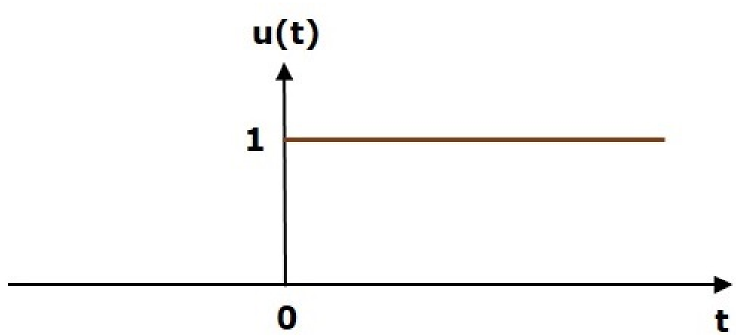



**Solution:**

To get the response (output) of the first order system in the time domain, we must follow these four steps:

1) Take the Laplace transform of the input signal **r(t)** in order to get **R(s)**.

A unit step signal **u(t)** is defined as: $r(t) = u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$




After applying Laplace transform on both the sides: $R(s) = \frac{1}{s}$.

2) Substitute **G(s)=1/sT** in the equation:

$$\frac{\boldsymbol{C(s)}}{\boldsymbol{R(s)}} = \frac{\boldsymbol{G(s)}}{\boldsymbol{1 + G(s)}}$$

Where $\frac{C(s)}{R(s)}$ is the transfer function of the closed loop control system with unity negative feedback.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

**3)** Substitute **R(s)=** $\frac{1}{s}$ in the above equation in order to get **C(s)**.

We can rewrite the above equation as:

$$C(s) = \left(\frac{1}{sT + 1}\right) R(s)$$

$$\therefore C(s) = \left(\frac{1}{sT + 1}\right)\left(\frac{1}{s}\right) = \frac{1}{s(sT + 1)}$$

**4)** Do partial fraction of **C(s)** if required, then apply inverse Laplace transform to **C(s)** in order to get **c(t)**.

$$C(s) = \frac{1}{s(sT + 1)} = \frac{A}{s} + \frac{B}{sT + 1} = \frac{A(sT + 1) + Bs}{s(sT + 1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT + 1) + Bs$$

$$1 = ATs + A + Bs$$

$$1 = A + (AT + B)s$$

By equating the constant terms on both the sides, you will get **A = 1**.

Substitute A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0 = AT + B$$

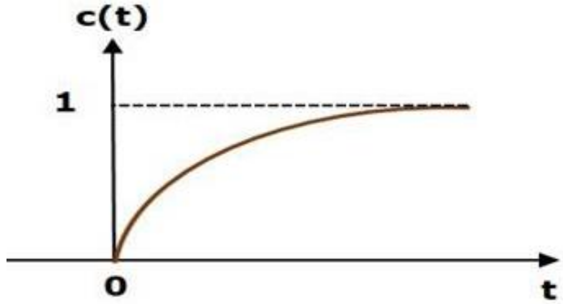
$$0 = (1)T + B \Rightarrow \boldsymbol{B = -T}$$

Substitute A = 1 and B = −T in partial fraction expansion of C(s).

$$C(s) = \frac{A}{s} + \frac{B}{sT+1} = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s+\frac{1}{T}\right)} = \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

Apply Inverse Laplace Transform on both the sides:

$$c(t) = \left(1 - e^{\left(-\frac{t}{T}\right)}\right)u(t)$$




$$Initial\ Value = \lim_{t\to 0} c(t)$$

$$= \lim_{t\to 0}\left(1 - e^{\left(-\frac{t}{T}\right)}\right)u(t) = \left(1 - e^{\left(-\frac{0}{T}\right)}\right)u(0)$$

$$= (1 - e^{0})u(0)$$

But $e^{0} = 1, and\ u(0) = 1$

$$\boldsymbol{Initial\ Value} = (1-1)(1) = \mathbf{0}$$

**Or**

$$\boldsymbol{Initial\ Value} = \lim_{s\to\infty} s\,C(s) = \lim_{s\to\infty} s\frac{1}{s(sT+1)} = \lim_{s\to\infty}\frac{1}{(sT+1)} = \frac{1}{((\infty)T+1)} = \frac{1}{\infty} = \mathbf{0}$$

$$Final\ Value = \lim_{t\to\infty} c(t) = \lim_{t\to\infty}\left(1 - e^{\left(-\frac{t}{T}\right)}\right)u(t) = \left(1 - e^{\left(-\frac{\infty}{T}\right)}\right)u(\infty) = \left(1 - e^{(-\infty)}\right)u(\infty)$$

But $e^{-\infty} = \frac{1}{e^{\infty}} = \frac{1}{\infty} = 0, and\ u(\infty) = 1$

$$\boldsymbol{Final\ Value} = (1-0)(1) = \mathbf{1}$$

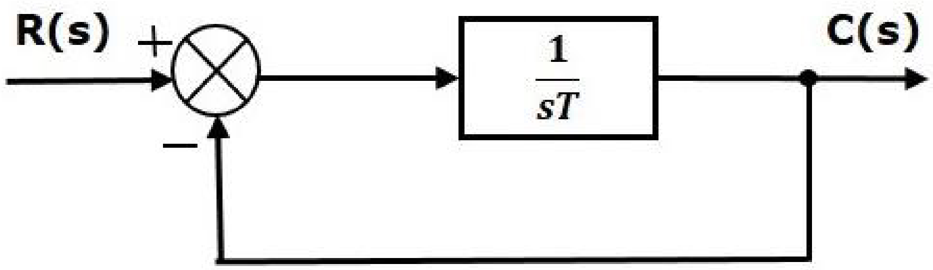
**Or**

$$\boldsymbol{Final\ Value} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s\frac{1}{s(sT+1)} = \lim_{s\to 0}\frac{1}{(sT+1)} = \frac{1}{((0)T+1)} = \frac{1}{1} = \mathbf{1}$$

$$\boldsymbol{e(t) = r(t) - c(t)} = u(t) - \left(1 - e^{\left(-\frac{t}{T}\right)}\right)u(t) = \boldsymbol{e^{\left(-\frac{t}{T}\right)}u(t)}$$

$$\boldsymbol{e_{ss}} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} e^{\left(-\frac{t}{T}\right)}u(t) = e^{\left(-\frac{\infty}{T}\right)}u(\infty) = \mathbf{0}$$

**Example 4: For the following block diagram of the closed loop control system, find the time response, initial value, final value, and steady-state error if the applied input is a unit ramp signal.**



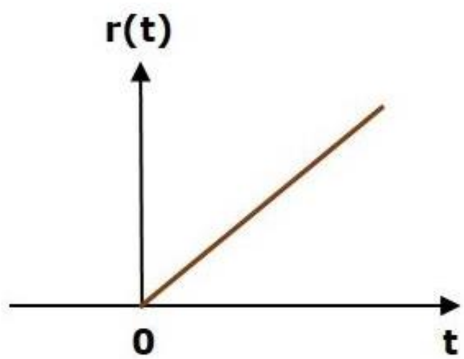

**Solution:**

To get the response (output) of the first order system in the time domain, we must follow these four steps:

**1)** Take the Laplace transform of the input signal **r(t)** in order to get **R(s)**.

A unit ramp signal **r (t)** is defined as:

$$\boldsymbol{r(t) = t\,u(t) = \begin{cases} t, & t \geq 0 \\ 0, & t < 0 \end{cases}}$$




Apply Laplace transform on both the sides: $\boldsymbol{R(s)} = \frac{\boldsymbol{1}}{\boldsymbol{s^2}}$.

**2)** Substitute **G(s)=1/sT** in the equation:

$$\boldsymbol{\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}}$$

Where $\boldsymbol{\frac{C(s)}{R(s)}}$ is the transfer function of the closed loop control system with unity negative feedback.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1 + \frac{1}{sT}} = \frac{1}{sT + 1}$$

**3)** Substitute **R(s)=** $\frac{1}{s^2}$ in the above equation in order to get **C(s)**.

We can rewrite the above equation as:

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s^2}\right) = \frac{1}{s^2(sT+1)}$$

**4)** Do partial fraction of **C(s)** if required, then apply inverse Laplace transform to **C(s)** in order to get **c(t)**.

$$C(s) = \frac{1}{s^2(sT+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1} = \frac{A(sT+1) + Bs(sT+1) + Cs^2}{s^2(sT+1)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A(sT+1) + Bs(sT+1) + Cs^2$$

$$1 = ATs + A + BTs^2 + Bs + Cs^2$$

$$1 = A + ATs + Bs + BTs^2 + Cs^2$$

$$1 = A + (AT+B)s + (BT+C)s^2$$

By equating the constant terms on both the sides, you will get **A = 1**.

Substitute A = 1 and equate the coefficient of the s terms on both the sides.

$$0 = AT + B$$

$$0 = (1)T + B \Rightarrow \boldsymbol{B = -T}$$

Similarly, substitute B = −T and equate the coefficient of $s^2$terms on both the sides. You will get:

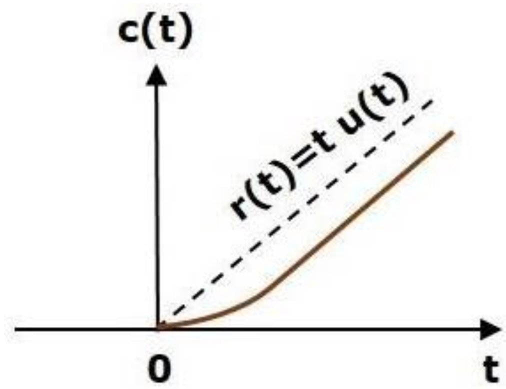
$$0 = BT + C \Longrightarrow 0 = (-T)T + C$$

$$0 = -T^2 + C \Longrightarrow \boldsymbol{C = T^2}$$

Substitute A = 1, B = −T and $C = T^2$ in the partial fraction expansion of C(s).

$$C(s) = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{sT+1} = \frac{1}{s^2} - \frac{T}{s} + \frac{T^2}{T(s+\frac{1}{T})} = \frac{1}{s^2} - \frac{T}{s} + \frac{T}{s+\frac{1}{T}}$$

Apply Inverse Laplace Transform on both the sides:




$$c(t) = \left(t - T + Te^{\left(-\frac{t}{T}\right)}\right)u(t)$$

$$Initial\ Value = \lim_{t\to 0} c(t)$$

$$= \lim_{t\to 0}\left(t - T + Te^{\left(-\frac{t}{T}\right)}\right)u(t)$$

$$= \left(0 - T + Te^{\left(-\frac{0}{T}\right)}\right)u(0)$$

But $e^0 = 1, and\ u(0) = 1$

$$\boldsymbol{Initial\ Value} = \big(0 - T + T(1)\big)(1) = -T + T = \mathbf{0}$$

**Or**

$$Initial\ Value = \lim_{s\to\infty} s\,C(s) = \lim_{s\to\infty} s\frac{1}{s^2(sT+1)} = \lim_{s\to\infty}\frac{1}{s(sT+1)} = \frac{1}{\infty((\infty)T+1)} = \frac{1}{\infty}$$

$$\boldsymbol{Initial\ Value} = \mathbf{0}$$

$$\boldsymbol{Final\ Value} = \lim_{t\to\infty} c(t) = \lim_{t\to\infty}\left(t - T + Te^{\left(-\frac{t}{T}\right)}\right)u(t) = \left(\infty - T + Te^{\left(-\frac{\infty}{T}\right)}\right)u(\infty) = \infty$$

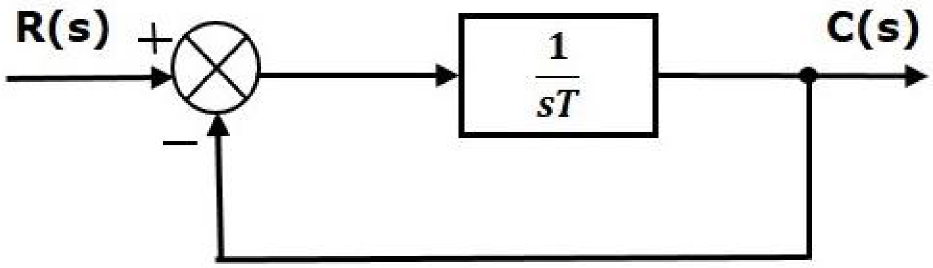
**Or**

$$\boldsymbol{Final\ Value} = \lim_{s\to 0} s\,C(s) = \lim_{s\to 0} s\frac{1}{s^2(sT+1)} = \lim_{s\to 0}\frac{1}{s(sT+1)} = \frac{1}{0((0)T+1)} = \frac{1}{0} = \infty$$

$$\boldsymbol{e(t) = r(t) - c(t)} = t\,u(t) - \left(t - T + Te^{\left(-\frac{t}{T}\right)}\right)u(t) = \boldsymbol{\left(T - Te^{\left(-\frac{t}{T}\right)}\right)u(t)}$$

$$\boldsymbol{e_{ss}} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty}\left(T - Te^{\left(-\frac{t}{T}\right)}\right)u(t) = \left(T - Te^{\left(-\frac{\infty}{T}\right)}\right)u(\infty) = (T - T(0))(1) = \boldsymbol{T}$$

**Homework 1: For the following block diagram of the closed loop control system, find the time response, initial value, final value, and steady-state error if the applied input is a unit impulse signal.**




**Homework 2:** **For the following block diagram of the closed loop control system, find the time response, initial value, final value, and steady-state error if the applied input is a unit parabolic signal.**